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Current-Algebra Sum Rules for States of Arbitrary Mass and Spin

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Sum rules based on current algebra are obtained for the currents evaluated between states of arbitrary spin and mass. Both the infinite-momentum limit and the dispersion method are shown to yield the same result. These sum rules are given explicitly for the crossed-channel amplitudes.

I. INTRODUCTION

SUM rules from current algebra have been obtained principally using two methods. One is to evaluate the commutators in the infinite-total-momentum limit.[1] The other is through the use of dispersion relations.[2] Explicit sum rules have been written for the case of scalar particles[1,2] and for equal-mass spin-one-half states.[3,4] The other sum rules have usually involved the evaluation of the commutators between states with equal momentum.

In this article sum rules for the commutator between arbitrary spin and mass states are obtained. Both methods have been shown to give the same results. The commutation relations are[5]

$$\begin{aligned}\delta(x_0)[J_V{}^{0;\,\alpha}(x), J_V{}^{\nu;\,\beta}(0)] &= ic^{\alpha\beta\gamma}\delta^4(x)J_V{}^{\nu;\,\gamma}(0)\,,\\ \delta(x_0)[J_V{}^{0;\,\alpha}(x), J_A{}^{\nu;\,\beta}(0)] &= ic^{\alpha\beta\gamma}\delta^4(x)J_A{}^{\nu;\,\gamma}(0)\,,\\ \delta(x_0)[J_A{}^{0;\,\alpha}(x), J_A{}^{\nu;\,\beta}(0)] &= ic^{\alpha\beta\gamma}\delta^4(x)J_V{}^{\nu;\,\gamma}(0)\,,\end{aligned} \tag{1a}$$

which we summarize as

$$\delta(x_0)[J^{0;\,\alpha}(x), J^{\nu;\,\beta}(0)] = if^{\alpha\beta\gamma}\delta^4(x)J^{\nu;\,\gamma}(0)\,. \tag{1b}$$

In Eq. (1a) α,β,γ run through the eight $SU(3)$ indices and the subscript V,A denotes vector or axial-vector currents. In Eq. (1b) the indices run through the $SU(3)\times SU(3)$ group. The $c^{\alpha\beta\gamma}$, $f^{\alpha\beta\gamma}$ are structure constants of $SU(3)$ and $SU(3)\otimes SU(3)$, respectively.

The above commutation relations are presumed correct for $\nu=0$. For $\nu=1,2,3$ there appear additional gradient Schwinger terms.[6] The existence of these terms does not come into the discussions below.

In Sec. II a general representation for these commutators, evaluated between helicity states of arbitrary mass and spin, is developed in terms of helicity amplitudes in the crossed channel. Dispersion relations for kinematically free amplitudes are stated.

In Sec. III the symmetry properties due to parity conservation for both the amplitudes and vertex functions are presented.

In Sec. IV sum rules are obtained and summarized in Sec. V for the cases of spin zero and one-half.

The use of the crossed-channel amplitudes directly will facilitate discussion of convergence based on Regge-pole arguments.

II. KINEMATIC ANALYSIS AND DISPERSION RELATIONS

We are interested in studying the matrix element of the commutator of two currents between arbitrary hadron states. We choose to label these states by their three-momenta, helicities, and a myriad of other quantum numbers which do not change under Lorentz transformations. These other quantum numbers will not be explicitly shown. Sum rules may be obtained from either of the following expressions[7]:

$$\begin{aligned}S_{\lambda_1\,\lambda_2}{}^{\mu\nu;\,\alpha\beta}(P,\Delta,Q) &= (2\pi)^3(4\omega_1\omega_2)^{1/2}\int d^4x\, e^{iq_1\cdot x}\theta(x_0)\\ &\quad\times\langle \mathbf{p}_1;\lambda_1|[J^{\mu;\,\alpha}(x), J^{\nu;\,\beta}(0)]|\mathbf{p}_2;\lambda_2\rangle\,,\end{aligned} \tag{2}$$

or

$$\begin{aligned}R_{\lambda_1\,\lambda_2}{}^{\mu\nu;\,\alpha\beta}(P,\Delta,Q) &= \frac{(2\pi)^3}{2i}(4\omega_1\omega_2)^{1/2}\int d^4x\, e^{iq_1\cdot x}\\ &\quad\times\langle \mathbf{p}_1;\lambda_1|[J^{\mu;\,\alpha}(x), J^{\nu;\,\beta}(0)]|\mathbf{p}_2;\lambda_2\rangle\,,\end{aligned} \tag{3}$$

which is related to the absorptive part of S. In the above

$$P=p_1+p_2,\quad \Delta=p_1-p_2,\quad Q=q_1+q_2\,,$$

with $q_1+p_1=q_2+p_2$ and $\omega_i=(p_i{}^2+m_i{}^2)^{1/2}$, where m_i is the invariant mass of state i. We likewise introduce the invariant variables

$$\nu=P\cdot Q,\quad t=\Delta^2.$$

Under the Lorentz group, S transforms as follows:

$$\begin{aligned}S_{\lambda_1,\lambda_2}{}^{\mu\nu;\,\alpha\beta}(P,\Delta,Q) &= \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}D_{\lambda_1'\lambda_2}{}^{*(J_1)}(W(\mathbf{p}_1,\Lambda))D_{\lambda_2'\lambda_2}{}^{(J_2)}(W(\mathbf{p}_2,\Lambda))\\ &\quad\times S_{\lambda_1',\lambda_2'}{}^{\mu'\nu';\,\alpha\beta}(\Lambda P,\Lambda\Delta,\Lambda Q)\,,\end{aligned} \tag{4}$$

with an analogous expression for R. In the above Λ is an arbitrary Lorentz transformation and $D_{\lambda',\lambda}{}^{(J)}(W(\mathbf{p},\Lambda))$ is the spin-J representation of the rotation group for the Wigner rotation determined by $\mathbf{p}$ and Λ.

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[1] R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1966).

[2] S. Fubini and G. Furlan, Physics **1**, 229 (1965); S. Fubini, Nuovo Cimento **43A**, 475 (1966).

[3] W. Weisberger, Phys. Rev. **143**, 1302 (1966); S. Adler, *ibid.* **143**, 1144 (1966); I. Muzinich, *ibid.* **151**, 1206 (1966); D. Amati, R. Jengo, and E. Remiddi (unpublished).

[4] J. Bjorken, Phys. Rev. **148**, 1467 (1966).

[5] M. Gell-Mann, Physics **1**, 63 (1962).

[6] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[7] $S^{\mu\nu}$ is not quite a covariant amplitude. There exists non-covariant parts which are related to the Schwinger terms. For a full discussion see Ref. 4.

S may be considered a scattering amplitude for a current of momentum q_1 impinging on a state with momentum p_1 and going to a current of momentum q_2 and a state with momentum p_2.

We shall obtain a representation for S and R in the Breit coordinate system, $\mathbf{P}=0$, $\mathbf{\Delta}$ along the z direction, and Q in the x-z plane. The form in any other frame of reference may be obtained by means of Eq. (4). Quantities in the Breit system will be denoted by the subscript B.

$$\begin{aligned} P_B{}^\mu &= \big((2m_1{}^2+2m_2{}^2-t)^{1/2};\,0;\,0;\,0\big),\\ \Delta_B{}^\mu &= \big((m_1{}^2-m_2{}^2)/\sqrt{(p^2)};\,T(m)/\sqrt{(p^2)};\,0;\,0\big),\\ Q_B{}^\mu &= \big(\nu/\sqrt{(p^2)};\,[q_1{}^2-q_2{}^2+\nu(m_1{}^2-m_2{}^2)/P^2]\\ &\qquad\times\sqrt{(p^2)}/T(m);\,\{(\nu^2/P^2)\\ &\qquad-[q_1{}^2-q_2{}^2+\nu(m_1{}^2-m_2{}^2)/P^2]^2\\ &\qquad\qquad\times P^2/T^2(m)-Q^2\}^{1/2};\,0\big), \end{aligned} \tag{5}$$

with

$$T(m)=[t^2-2t(m_1{}^2+m_2{}^2)+(m_1{}^2-m_2{}^2)^2]^{1/2}.$$

It is straightforward to show that the Lorentz transformation needed to take us from the Breit system to an arbitrary one specified by P,Δ,Q is given by

$$\begin{aligned} \Lambda^\mu{}_0 &= P^\mu/P_B{}^0,\\ \Lambda^\mu{}_z &= (\Delta^\mu-\Lambda^\mu{}_0\Delta_B{}^0)/\Delta_B{}^z,\\ \Lambda^\mu{}_x &= (Q^\mu-\Lambda^\mu{}_0 Q_B{}^0-\Lambda^\mu z Q_B{}^z)/Q_B{}^x,\\ \Lambda^\mu{}_y &= \epsilon^\mu{}_{\alpha\beta\nu}P^\alpha\Delta^\beta Q^\nu/\sqrt{(\xi^2)}, \end{aligned} \tag{6}$$

with $\xi^2=[Q_B{}^xT(m)]^2$. It is convenient to write Q^μ as

$$Q^\mu=\Lambda^\mu{}_0Q_B{}^0+\Lambda^\mu{}_zQ_B{}^z+K^\mu,$$

with $K\cdot P=K\cdot\Delta=0$ and $K^2=-(Q_B{}^x)^2$. $\Lambda^\mu{}_x$ and $\Lambda^\mu{}_y$ may then be rewritten

$$\begin{aligned} \Lambda^\mu{}_x &= K^\mu/Q_B{}^x,\\ \Lambda^\mu{}_y &= \epsilon^\mu{}_{\alpha\beta\gamma}P^\alpha\Delta^\beta K^\gamma/\sqrt{(\xi^2)} \end{aligned} \tag{6$'$}$$

which shows their independence of ν.

The convenience of the Breit system is that the amplitude S_B is simply related to the crossed-channel amplitude for the reaction $(p_1,\lambda_1)+(\bar{p}_2,\lambda_2)\rightarrow(\bar{q}_1,\,\mu;\,\bar{\alpha})+(q_2,\,\nu;\,\beta)$, where the bars denote antiparticles and $\bar{\alpha}$ denotes the complex-conjugate current. If $\bar{S}$ denotes the crossed-channel amplitude, then following the arguments of Trueman and Wick,[8] we find that S_B equals $\bar{S}$ in the system $p_1=\bar{p}_2$ with center-of-mass energy $\sqrt{t}$ and momentum transfer $\frac{1}{2}(\nu+m_1{}^2+m_2{}^2+q_1{}^2+q_2{}^2-t)$. To relate S_B to $\bar{S}$ in the crossed-channel center-of-mass system we must perform a Lorentz transformation along the z direction with velocity $T(m)/(m_1{}^2-m_2{}^2)$. As this is a Lorentz transformation along the z direction the helicities can either remain the same or change signs. It turns out that λ_2 changes signs.[9] If L is the above-mentioned transformation, then

$$\bar{S}_{B\lambda_1,\lambda_2}{}^{\mu\nu;\;\bar{\alpha}\beta}=L^\mu{}_{\mu'}L^\nu{}_{\nu'}\bar{S}_{\lambda_1,-\lambda_2}{}^{\mu;\;\nu';\;\bar{\alpha}\beta}. \tag{7}$$

Each current may be considered a superposition of a vector and a scalar "particle." (For conserved currents the scalar part is missing from the absorptive part.) We wish to decompose the amplitude $\bar{S}$ into helicity amplitudes. The reason is that we are easily able to remove kinematic ν singularities from $\bar{S}$ helicity amplitudes.[10,11]

We define the crossed-channel amplitudes by

$$\begin{aligned} \bar{S}_{\mathrm{c.m.}\lambda_1,\lambda_2}{}^{\mu\nu;\;\bar{\alpha}\beta} &= \sum_{(l_1,\sigma_1)}\sum_{(l_2\sigma_2)} t^\mu(\bar{q}_1;\,l_1,\,\sigma_1)t^\nu(q_2;\,l_2,\,\sigma_2)\\ &\qquad\times A^{\bar{\alpha}\beta}{}_{(l_1,\sigma_1),(l_2,\sigma_2);\;\lambda_1,\lambda_2}(\nu,t), \end{aligned} \tag{8}$$

where the summation is over $(l,\sigma)=(0,0)$, $(1,+)$, $(1,0)$, and $(1,-)$, and $t^\mu(q;\,l,\,\sigma)$ may be determined as follows: Let $t^\mu{}_\nu(q)$ be the Lorentz transformation which takes the vector $(\sqrt{(q^2)};\,0;\,0;\,0)$ to q^μ. For q spacelike, $t^\mu{}_\nu(q)$ is complex. The transformation is further specified by requiring that it be a boost in the z direction followed by a rotation in the $\mathbf{q}-z$ plane. For $\mathbf{q}$ in the x-z plane

$$t^\mu{}_\nu(q)=\frac{1}{\sqrt{(q^2)}}\begin{array}{c}\begin{array}{cccc}\nu=0 & 1 & 2 & 3\end{array}\\ \left(\begin{array}{cccc} q^0 & [(q^0)^2-q^2]^{1/2} & 0 & 0\\ q^z & q^zq^0/[(q^0)^2-q^2]^{1/2} & -q^x\sqrt{(q^2)}/[(q^0)^2-q^2]^{1/2} & 0\\ q^x & q^xq^0/[(q^0)^2-q^2]^{1/2} & q^z\sqrt{(q^2)}/[(q^0)^2-q^2]^{1/2} & 0\\ 0 & 0 & 0 & 1 \end{array}\right)\end{array}\begin{array}{c}\mu=0\\1\\2\\3\end{array}. \tag{9}$$

Then for Eq. (8)

$$\begin{aligned} t^\mu(q;\,0,\,0) &= t^\mu{}_0(q),\\ t^\mu(q;\,1,\,+) &= -[t^\mu{}_x(q)+it^\mu{}_y(q)]/\sqrt{2},\\ t^\mu(q;\,1,\,0) &= t^\mu{}_z/(q),\\ t^\mu(q;\,1,\,-) &= [t^\mu{}_x(q)-it^\mu{}_y(q)]/\sqrt{2}. \end{aligned} \tag{10}$$

Combining Eqs. (4), (5), (8), and (10) we find a representation for S in an arbitrary coordinate system in terms of the crossed-channel helicity amplitudes:

$$\begin{aligned} S_{\lambda_1\lambda_2}{}^{\mu\nu;\;\alpha\beta}(P,\Delta,Q) &= D_{\lambda_1'\lambda_1}{}^{*(J_1)}(W(\mathbf{p}_1,\Lambda))D_{\lambda_2'\lambda_2}{}^{(J_2)}(W(\mathbf{p}_2,\Lambda)\\ &\quad\times\sum_{(l_1,\sigma_1)}\sum_{(l_2,\sigma_2)} Z^\mu(\bar{q}_1;\,l_1,\,\sigma_1)Z^\nu(q_2;\,l_2,\,\sigma_2)\\ &\qquad\times A^{\bar{\alpha},\beta}{}_{(l_1,\sigma_1)(l_2,\sigma_2);\;\lambda_1',\,\lambda_2'}(\nu,t), \end{aligned} \tag{11}$$

[8] T. Trueman and G. Wick, Ann. Phys. (N.Y.) **26**, 322 (1966); I. Muzinich, J. Math. Phys. **5**, 1418 (1964).

[9] The easiest way to see this is to take the crossing relations derived in Ref. 8 for direct-channel backward scattering where the Breit and center-of-mass systems coincide.

[10] L. C. Wang, Phys. Rev. **142**, 1187 (1966).

[11] Such a dispersion relation may be proved for $Q^2<0$ using the Jost-Lehman-Dyson representation.

where $Z^\mu(q;l,\sigma)$ is related to $Z^\mu{}_\nu(q)$ in the same way as $t^\mu(q;l,\sigma)$ is related to $t^\mu{}_\nu(q)$ [cf. Eq. (10)], and

$$Z^\mu{}_\nu(q)=[\Lambda Lt(q)]^\mu{}_\nu.$$

An elementary but tedious calculation yields

$$\begin{aligned}
Z^\mu{}_0(\bar q_1)&=(\Delta-Q)^\mu/2\sqrt{(q_1^2)}\,,\\
Z^\mu{}_z(\bar q_1)&=[(\Delta-Q)^\mu(t+q_1^2-q_2^2)-4q_1^2\Delta^\mu]/2\sqrt{(q_1^2)}T(q)\,,\\
Z^\mu{}_x(\bar q_1)&=\frac{T(q)}{T(m)}\frac{1}{4q_1^2Q_B{}^x}\{(\Delta^\mu Q)^\mu(m_1^2-m_2^2-\nu)\\
&\quad+2Z^\mu{}_z\sqrt{(q_1^2)}(\nu(t+q_1^2-q_2^2)\\
&\quad-(m_1^2-m_2^2)(t-3q_1^2-q_2^2))/\\
&\qquad T(q)-4q_1^2P^\mu\}\,,\\
Z^\mu{}_y(\bar q_1)&=\Lambda^\mu{}_y\,,
\end{aligned}\tag{12}$$

with $Q_B{}^x$ defined in Eq. (5), and

$$T(q)=[t^2-2t(q_1^2+q_2^2)+(q_1^2-q_2^2)]^{1/2}.$$

$Z^\mu{}_\nu(q_2)$ is obtained from $Z^\mu{}_\nu(\bar q_1)$ by interchanging q_1^2 and q_2^2, letting $\nu\to-\nu$, and $Q_B{}^x\to-Q_B{}^x$. A similar analysis may be carried out for R and we write the result

$$\begin{aligned}
R_{\lambda_1\lambda_2}{}^{\mu\nu;\,\alpha\beta}(P,\Delta,Q)
&=D_{\lambda_1',\lambda_1}{}^{*(J_1)}(W(\mathbf{p}_1,\Lambda))D_{\lambda_1',\lambda_2'}{}^{(J_2)}(W(\mathbf{p}_2,\Lambda))\,,\\
&\times\sum_{(l_1,\sigma_1)}\sum_{(l_2,\sigma_2)}Z^\mu(\bar q_1;l_1\sigma_1)Z^\nu(q_2;l_2,\sigma_2)\\
&\qquad\times a^{\alpha,\beta}{}_{(l_1,\sigma_1)(l_2,\sigma_2);\,\lambda_1',-\lambda_2'}(\nu,t)\,.
\end{aligned}\tag{13}$$

From Ref. 10 we know that

$$A^{\bar\alpha\beta}{}_{(l_1\sigma_1),\,(l_2,\sigma_2);\,\lambda_1,\lambda_2}(\nu,t)/[\cos\tfrac12\theta_{t/2}]^{|\lambda+\sigma|}[\sin\tfrac12\theta_t]^{|\lambda-\sigma|}$$

has no kinematic ν singularities. In the above $\lambda=\lambda_1-\lambda_2$ and $\sigma=\sigma_1-\sigma_2$ and θ_t is the t-channel scattering angle. The expression for the angle may be obtained from

$$\begin{aligned}
\cos\theta_t(\nu)&=-[\nu t+(q_1^2-q_2^2)(m_1^2-m_2^2)]/\\
&\qquad[T(m)T(q)]\,,\\
\sin\theta_t(\nu)&=-Q_B{}^x\sqrt{t}/T(q)\,.
\end{aligned}\tag{14}$$

We assume that the A's satisfy the following dispersion relation, which we write in unsubtracted form:

$$\begin{aligned}
A^{\alpha\beta}{}_{(l_1\sigma_1),(l_2,\sigma_2);\,\lambda_1,\lambda_2}(\nu,t)
&=\frac{[\cos\frac12\theta_t(\nu)]^{|\lambda+\sigma|}[\sin\frac12\theta_t(\nu)]^{|\lambda-\sigma|}}{\pi}\\
&\times\int\frac{a_{(l_1,\sigma_1),(l_2,\sigma_2);\,\lambda_1,\lambda_2}(\nu',t)d\nu'}{[\cos\frac12\theta_t(\nu')]^{|\lambda+\sigma|}[\mathrm{sm}\frac12\theta_t(\nu')]^{|\lambda-\sigma|}(\nu'-\nu)}\,.
\end{aligned}\tag{15}$$

A representation for the vertex function will also be required. This has been given by Durand, DeCelles, and Marr.[12] Let

$$\Gamma_{\lambda_1,\lambda_2}{}^{\mu;\,\gamma}(P,\Delta)=(2\pi)^3(4\omega_1\omega_2)^{1/2}\langle\mathbf{p}_1;\lambda_1|J^{\mu;\,\gamma}(0)|\mathbf{p}_2;\lambda_2\rangle\,,$$

then for any Lorentz transformation

$$\begin{aligned}
\Gamma_{\lambda_1\lambda_2}{}^{\mu;\,\gamma}(P,\Delta)&=\Lambda^\mu{}_{\mu'}D_{\lambda_1',\lambda_1}{}^{*(J_1)}(W(\mathbf{p}_1,\Lambda))\\
&\times D_{\lambda_1',\lambda_1}{}^{(J_2)}(W(\mathbf{p}_2,\Lambda))\Gamma_{\lambda_1',\lambda_2'}{}^{\mu^1;\,\gamma}(\Lambda P,\Lambda Q)\,,
\end{aligned}\tag{16}$$

and it suffices to know Γ in the Breit system, where we denote it by $\Gamma_{\lambda_1,\lambda_2}{}^{\mu;\,\gamma}(t)$. From Ref. (12) we know that due to rotational invariance

$$\begin{aligned}
\Gamma_{\lambda_1,\lambda_2}{}^{0;\,\gamma}(t)&=\delta_{\lambda_1,-\lambda_2}\Gamma_{\lambda_1,\lambda_2}{}^{0;\,\gamma}(t)\,,\\
\Gamma_{\lambda_1,\lambda_2}{}^{z;\,\gamma}(t)&=\delta_{\lambda_1,\,-\lambda_2}\Gamma_{\lambda_1,\lambda_2}{}^{z,\gamma}(t)\,,\\
\Gamma_{\lambda_1,\lambda_2}{}^{\pm;\,\gamma}(t)&=\delta_{\lambda_1-\lambda_2\pm1}\Gamma_{\lambda_1,\,\lambda_2}{}^{\pm;\,\gamma}(t)\,,
\end{aligned}$$

with $\Gamma^+=-(\Gamma^x+i\Gamma^y)/\sqrt2$ and $\Gamma^-=(\Gamma^x-i\Gamma^y)/\sqrt2$.

III. SYMMETRY PROPERTIES

Unless the states under consideration are identical and eigenstates of the scattering-matrix, time-reversal invariance does not lead to any interesting restrictions. The only useful relations among the S's, R's, and Γ's come from parity invariance.

Let η_1,η_2 be the intrinsic parities of the states; η_α,η_β the parities of the vector parts of the currents (i.e., minus for vector current and plus for pseudovector currents). Then

$$\begin{aligned}
A^{\alpha\beta}{}_{(l_1,\sigma_1),(l_2,\sigma_2)\lambda_1,\lambda_2}(\nu,t)
&=\frac{\eta_1\eta_2}{\eta_\alpha\eta_\beta}(-1)^{J_1+J_1+l_1+l_2}(-1)^{-\lambda+\lambda'+\sigma+\sigma'}\\
&\qquad\times A^{\alpha\bar\beta}{}_{(l_1,\,-\sigma)(l_2,\,-\sigma_2);\,-\lambda_1,\,-\lambda_2}(\nu,t)\,.
\end{aligned}\tag{17}$$

The above expression takes into account the fact that if 2 is a Fermi state, then in the crossed channel, the charge-conjugate state has opposite parity. An analogous expression holds for the a's.

For the vertex function, parity conservation implies

$$\Gamma_{\lambda_1,\lambda_2}{}^{0,\,x,\,z;\,\gamma}(t)=\frac{\eta_1\eta_2}{\eta_\gamma}(-1)^{J_1+J_2-\lambda_1-\lambda_2}\Gamma^{0,\,x,\,z;\,\gamma}(t)\,,\tag{18}$$

with an extra minus sign for Γ^y.

In the subsequent discussion it will be convenient to introduce "parity eigenstates"[13]

$$|\mathbf{p};\lambda\pm\rangle=(1/\sqrt2)(|\mathbf{p};\lambda\rangle\pm|\mathbf{p};-\lambda\rangle)\tag{19}$$

for $\lambda=0,1\cdots J$, and to express the amplitudes and vertex in terms of these states.

IV. SUM RULES

A. $P\to\infty$ Method

Using[1] translation invariance we rewrite Eq. (3) as

$$\begin{aligned}
R_{\lambda_1\lambda_2}{}^{\mu\nu;\,\alpha\beta}(P,\Delta,Q)
&=\frac{(2\pi)^3}{2i}(4\omega_1\omega_2)^{1/2}\int d^4x\,e^{\frac12iQ\cdot x}\\
&\qquad\times\langle\mathbf{p}_1;\lambda_1|[J^{\mu;\,\alpha}(\tfrac12x),J^{\nu;\,\beta}(-\tfrac12x)|\mathbf{p}_2;\lambda_2\rangle
\end{aligned}\tag{20}$$

[12] L. Durand, D. De Celles, and R. Marr, Phys. Rev. **126**, 1882 (1962).

[13] These are really eigenstates of $Y=Pe^{i\pi J_2}$. (See Ref. 12.)

and note that using the commutation relations, Eq. (1),

$$\int R_{\lambda_1,\lambda_2}{}^{00;\ \alpha\beta}(P,\Delta,Q)dQ_0|_{\mathbf{Q}\text{ fixed}}$$
$$=(4\omega_1\omega_2)^{1/2}\frac{(2\pi)^4}{i}\int d^4x\, e^{iQ\cdot x/2}\delta(x_0)$$
$$\times\langle\mathbf{p}_1;\lambda_1|[J^{0;\ \alpha}(x/2),J^{0;\ \beta}(-x/2)]|\mathbf{p}_2;\lambda_2\rangle$$
$$=2\pi f^{\alpha\gamma\beta}\Gamma^{0;\ \gamma}(P,\Delta).\quad(21)$$

Since $\mathbf{Q}$ is fixed in the above, both q_1^2 and q_2^2 vary as we integrate along Q_0. To get sum rules for fixed q_1^2 and q_2^2 we choose a particular frame. First perform an acceleration along the z direction to eliminate the time component of Δ^μ and then accelerate to $\pm\infty$ along the y direction. Note that any Wigner rotations associated with the foregoing transformations are the same on both sides of Eq. (21) and hence may be ignored. Changing variables from Q_0 to $\nu=Q_0P_0-\mathbf{Q}\cdot\mathbf{P}$ we find that in the infinite-momentum limit (keeping ν fixed)[14]

$$Q^2\rightarrow-\mathbf{Q}^2,\quad(22)$$
$$\epsilon^0{}_{\alpha\beta\gamma}P^\alpha\Delta^\beta K^\gamma/\surd(\xi^2)\rightarrow p^y\surd(-t)/T(m).$$

The crucial assumption is that for the leading terms in P^0, P^y the limit is uniform. (This is equivalent to the assumption that certain terms in the expansion of $S^{\mu\nu}$ obey unsubtracted dispersion relations). Combining Eqs. (11), (12), (16), and (22) we obtain two sets of sum rules:

$$\frac{t}{T^2(m)}\int\frac{d\nu}{\sin^2\theta_t(\nu)}\{a^{\bar{\alpha}\beta}{}_{(1,+),\ (1,-);\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)$$
$$+a^{\bar{\alpha}\beta}{}_{(1,-),\ (1,+);\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)\}$$
$$=\frac{2\pi f^{\alpha\beta\gamma}}{(2m_1^2+2m_2^2-t)^{1/2}}\left\{\Gamma_{\lambda_1,\lambda_2}{}^{0;\ \gamma}(t)-\frac{(m_1^2-m_2^2)}{T(m)}\Gamma_{\lambda_1,\lambda_2}{}^{z;\ \gamma}(t)\right\},\quad(23)$$

and

$$\frac{\surd t}{(Tm)}\int\frac{d\nu}{\sin\theta_t(\nu)}\{A^{\bar{\alpha}\beta}{}_{(1,-),\ (1,+);\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)$$
$$-a^{\bar{\alpha},\beta}{}_{(1,-1),\ (1,+1);\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)\}$$
$$=\sqrt{2}\pi f^{\alpha\beta\gamma}[\Gamma_{\lambda_1,\lambda_2}{}^{+;\ \gamma}(t)+\Gamma_{\lambda_1,\lambda_2}{}^{-;\ \gamma}(t)].\quad(24)$$

If these sum rules are taken between parity states [cf. Eq. (19)], then for each choice of λ_1,λ_2 either (23) or (24) is trivially satisfied by having both sides vanish.

[14] We must remember that in this method $\mathbf{Q}$ is fixed and unrelated to ν.

B. Dispersion-Relation Method

Starting[2] with Eq. (2) we evaluate, using integration by parts,

$$(q_1)_\mu S_{\lambda_1,\lambda_2}{}^{\mu\nu;\ \alpha\beta}(P,\Delta,Q)$$
$$=i(2\pi)^3(4\omega_1\omega_2)^{1/2}\int d^4e^{iq;\ x}\theta(x_0)$$
$$\times\langle\mathbf{p}_1;\lambda_1|[D^\alpha(x),J^{\nu;\ \beta}(0)]|\mathbf{p}_2;\lambda_2\rangle$$
$$-f^{\alpha\beta\gamma}\Gamma_{\lambda_1,\lambda_2}{}^{\nu;\ \gamma}(t)+\text{Schwinger terms}\quad(25)$$

(see Ref. 6), where $D^\alpha(x)=\partial_\mu J^{\mu;\ \alpha}(x)$. The first term on the right in Eq. (25) could likewise be analyzed by the methods discussed previously; however, the only assumption we need make are that terms proportional to P^ν and $\epsilon^\nu{}_{\alpha\beta\nu}P^\alpha\Delta^\beta Q^\gamma/\surd(\xi^2)$ vanish as $\nu\rightarrow\pm\infty$. Keeping terms proportional to the above we find two limiting relations:

$$\lim_{\nu\rightarrow\pm\infty}(\surd(q_1^2)A^{\bar{\alpha}\beta}{}_{(0,0),\ x;\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)\frac{T(q)}{T(m)Q_B{}^x}$$
$$=\frac{f^{\alpha\beta\gamma}}{(2m_1^2+2m_2^2-t)^{1/2}}\left[\Gamma_{\lambda_1,\lambda_2}{}^{0;\ \mu}(t)-\frac{-(m_1^2-m_2^2)}{T(m)}\times\Gamma_{\lambda_1,\lambda_2}{}^{z;\ \gamma}(t)+\frac{\surd t}{T(m)}\Gamma_{\lambda_1,\lambda_2}{}^{x;\ \gamma}(t)\right],\quad(26)$$

and

$$\lim_{\nu\rightarrow\pm\infty}(q_1^2)^{1/2}A^{\bar{\alpha}\beta}{}_{(0,0);\ y;\ \lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)$$
$$=f^{\alpha\beta\gamma}\Gamma_{\lambda_1,\lambda_2}{}^{y;\ \gamma}(t),\quad(27)$$

where the indices x,y denote the combinations $[(-)-(+)]/\sqrt{2}$ and $[(+)+(-)]/i\sqrt{2}$, respectively.

To obtain sum rules we must require that certain terms in the expansion of S satisfy unsubstracted dispersion relation, i.e., vanish as $\nu\rightarrow\pm\infty$. This requirement, applied to the coefficients of $(\Delta-Q)^\mu\epsilon^\nu{}_{\alpha\beta\gamma}P^\alpha\Delta^\beta Q^\gamma/\surd(\xi^2)$ (where we treat $Z^\mu{}_0$ and $Z^\mu{}_z$ as independent vectors), yields

$$\lim_{\nu\rightarrow\pm\infty}\frac{T(q)}{T(m)}\frac{1}{Q_B{}^x}A^{\bar{\alpha}\beta}{}_{(0,0),\ x;\ \lambda_1,\lambda_2}$$
$$=\pm\nu\frac{T^2(q)}{T^2(m)(Q_B{}^x)^2}\frac{1}{2q_1^2}\times\{A^{\bar{\alpha}\beta}{}_{x,x;\ \lambda_1\lambda_2}+A^{\bar{\alpha}\beta}{}_{y,y;\ \lambda_1\lambda_2}\},$$
$$\lim_{\nu\rightarrow\pm\infty}A^{\bar{\alpha},\beta}{}_{(0,0),y;\ \lambda_1,\ \lambda_2}=\frac{\nu T(q)}{T(m)}\frac{1}{Q_B{}^x}A^{\bar{\alpha}\beta}{}_{x,\ y;\ \lambda_1\lambda_2}.\quad(28)$$

Similar relations may be obtained by considering the quantity $S^{\mu\nu}(q_2)_\nu$ and combining the two sets of limiting relations we find

$$\lim_{\nu\to\pm\infty}\frac{t}{T^2(m)}\frac{1}{\sin^2\theta_t(\nu)}\times\{A^{\alpha\beta}{}_{(1,+),(1-);\lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)+A^{\bar{\alpha}\beta}{}_{(1-),(1,+);\lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)\}=\frac{-2f^{\alpha\beta\gamma}}{(2m_1^2+2m_2^2-t)^{1/2}}\left\{\Gamma_{\lambda_1,\lambda_2}{}^{0;\,\gamma}(t)-\frac{(m_1^2-m_2^2)}{T(m)}\Gamma_{\lambda_1,\lambda_2}{}^{z;\,\gamma}(t)\right\},\quad(29)$$

and

$$\lim_{\nu\to\pm\infty}\frac{\sqrt{t}}{T(m)}\frac{1}{\sin\theta_t(\nu)}\times\times\{A^{\bar{\alpha}\beta}{}_{(1,-),(1,+);\lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)-A^{\bar{\alpha}\beta}{}_{(1,+)(1,-);\lambda_1,-\lambda_2}(\nu,t;q_1^2,q_2^2)\}=-\sqrt{2}f^{\alpha\beta\gamma}[\Gamma_{\lambda_1,\lambda_2}{}^{+;\,\gamma}(t)+\Gamma_{\lambda_1,\lambda_2}{}^{-;\,\gamma}(t)].\quad(30)$$

Any relations other than the above which might have been obtained before combining expressions from $(q_1)_\mu S^{\mu\nu}$ and $S^{\mu\nu}(q_2)_\nu$ can be seen to be trivially satisfied because of parity invariance.

Equations (29) and (30) are not yet singularity free; however, noting that if the limit of the following expression exists,

$$\lim_{\nu\to\pm\infty}\frac{\nu\sqrt{P(\nu)}}{\pi}\int\frac{\alpha(\nu')}{(\nu'-\nu)[P(\nu')]^{1/2}}d\nu'=\text{const.},$$

(where $P(\nu)$ is a polynomial in ν), then

$$\frac{1}{\pi}\int\alpha(\nu')d\nu'=\text{const.}$$

Using the dispersion relations [Eq. (15)] and the above lemma we recover the sum rules of Eqs. (23) and (24). Besides the sum rules there also occur many relations with zero on the right-hand side. These are of the superconvergence type[15] and will not be further discussed.

V. SUMMARY

We restate the sum rules derived for the two practical cases of spin-zero and spin-one-half states:

Spin zero:

$$\frac{t}{T^2(m)}\int\frac{d\nu}{\sin^2\theta_t(\nu)}a^{\bar{\alpha}\beta}{}_{(1,+),(1,-)}(\nu,t;q_1^2,q_2^2)=\frac{\pi}{(2m_1^2+2m_2^2-t)^{1\,2}}f^{\alpha\beta\gamma}\times\left\{\Gamma^{0;\,\gamma}(t)-\frac{(m_1^2-m_2^2)}{T(m)}\Gamma^{z;\,\gamma}(t)\right\}=\pi f^{\alpha\beta\gamma}F_1{}^\gamma(t),\quad(31)$$

where $F_1{}^\gamma(t)$ is the form factor appearing in the tensor decomposition of the current,

$$\langle\mathbf{p}_1|J^{\mu;\,\gamma}|\mathbf{p}_2\rangle=(2\pi)^3(4\omega_1\omega_2)^{1/2}[F_1{}^\gamma(t)P^\mu+F_2{}^\gamma(t)\Delta^\mu].$$

Spin one-half:

$$\frac{t}{T^2(m)}\int\frac{d\nu}{\sin^2\theta_t(\nu)}\{a^{\bar{\alpha}\beta}{}_{(1,+),(1,-);1/2,1/2}\times(\nu,t;q_1^2;q_2^2)+a^{\bar{\alpha}\beta}{}_{(1,-),(1,+);1/2,1/2}(\nu,t;q_1^2,q_2^2)\}=\frac{2\pi}{(2m_1^2+2m_2^2-t)^{1/2}}\left\{\Gamma^{0;\,\gamma}(t)-\frac{m_1^2-m_2^2}{T(m)}\Gamma^{z;\,\gamma}(t)\right\},\quad(32)$$

and

$$\frac{\sqrt{t}}{T(m)}\int\frac{d\nu}{\sin\theta_t(\nu)}\{A^{\bar{\alpha}\beta}{}_{(1,-)(1,+);+1/2,-1/2}\times(\nu,t;q_1^2,q_2^2)-A^{\bar{\alpha}\beta}{}_{(1,-)(1,+);+1/2,-1/2}(\nu,t;q_1^2,q_2^2)\}=\sqrt{2}\pi f^{\alpha\beta\gamma}[\Gamma_{+1/2,\,1/2}{}^{+;\,\gamma}(t)+\Gamma_{+1/2,\,1/2}{}^{-;\,\gamma}(t)].\quad(33)$$

For equal mass we may identify the Γ's with the usual form factors. For vector current,

$$\Gamma_{1/2,\,-1/2}{}^{0;\,\gamma}(t)=2mG_E(t),$$

$$\frac{\Gamma_{+1/2,\,1/2}{}^{+;\,\gamma}+\Gamma_{+1/2,\,1/2}{}^{-;\,\gamma}}{\sqrt{2}i}=i(\sqrt{-t})G_M(t).$$

Other cases may be obtained using parity-transformation properties.

For higher spin states, sum rules for $\lambda_1\neq-\lambda_2$ and $\lambda_1\neq-\lambda_2\pm1$ have zero on the right-hand side and are of the superconvergence type.

[15] V. de Alfaro, S. Fubini, G. Furland, and G. Rosetti, Phys. Letters 21, 576 (1966).